# ON THE GROUP OF STRONG
# SYMPLECTIC HOMEOMORHISMS

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Abstract. We generalize the "hamiltonian topology" on hamiltonian isotopies to an intrinsic "symplectic topology" on the space of symplectic isotopies. We use it to define the group $SSympeo(M,\omega)$ of strong symplectic homeomorphisms, which generalizes the group $Hameo(M,\omega)$ of hamiltonian homeomorphisms introduced by Oh and Muller. The group $SSympeo(M,\omega)$ is arcwise connected, is contained in the identity component of $Sympeo(M,\omega)$; it contains $Hameo(M,\omega)$ as a normal subgroup and coincides with it when $M$ is simply connected. Finally its commutator subgroup $[SSympeo(M,\omega), SSympeo(M,\omega)]$ is contained in $Hameo(M,\omega)$.

## 1. Introduction

The Eliashberg-Gromov symplectic rigidity theorem says that the group $Symp(M,\omega)$ of symplectomorphisms of a closed symplectic manifold $(M,\omega)$ is $C^0$ closed in the group $Diff^\infty(M)$ of $C^\infty$ diffeomorphisms of $M$ ( see [8]). This means that the "symplectic" nature of a sequence of symplectomorphisms survives topological limits. Also Lalonde-McDuff-Polterovich have shown in [9] that for a symplectomorphism, being "hamiltonian" is topological in nature. These phenomenons attest that there is a $C^0$ *symplectic topology* underlying the symplectic geometry of a symplectic manifold $(M,\omega)$.

According to Oh-Muller ([10]), the automorphism group of the $C^0$ symplectic topology is the closure of the group $Symp(M,\omega)$ in the group $Homeo(M)$ of homeomorphisms of $M$ endowed with the $C^0$ topology. That group, denoted $Sympeo(M,\omega)$ has been called the group of symplectic homeomorphisms:

$$Sympeo(M,\omega) =: \overline{Symp(M,\omega)}.$$

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The $C^0$ topology on $Homeo(M)$ coincides with the metric topology coming from the metric

$$\overline{d}(g,h) = max(sup_{x \in M} d_0(g(x), h(x)), sup_{x \in M} d_0(g^{-1}(x), h^{-1}(x)))$$

where $d_0$ is a distance on $M$ induced by some riemannian metric [11].

On the space $PHomeo(M)$ of continuous paths $\gamma : [0,1] \to Homeo(M)$, one has the distance

$$\overline{d}(\gamma, \mu) = sup_{t \in [0,1]} \overline{d}(\gamma(t), \mu(t)).$$

Consider the space $PHam(M)$ of all isotopies $\Phi_H = [t \mapsto \Phi_H^t]$ where $\Phi_H^t$ is the family of hamiltonian diffeomorphisms obtained by integration of the family of vector fields $X_H$ for a smooth family $H(x,t)$ of real functions on $M$, i.e.

$$\frac{d}{dt} \Phi_H^t(x) = X_H(\Phi_H^t(x))$$

and $\Phi_H^0 = id$.

Recall that $X_H$ is uniquely defined by the equation

$$i(X_H)\omega = dH$$

where $i(.)$ is the interior product.

The set of time one maps of all hamiltonian isotopies $\{\Phi_H^t\}$ form a group, denoted $Ham(M, \omega)$ and called the group of hamiltonian diffeomorphisms.

**Definition** *The hamiltonian topology* [11] *on $PHam(M)$ is the metric topology* defined by the distance

$$d_{ham}(\Phi_H, \Phi_{H'}) = ||H - H'|| + \overline{d}(\Phi_H, \Phi_{H'})$$

where

$$||H - H'|| = \int_0^1 osc(H - H')dt.$$

and the oscillation of a function $u$ is

$$osc(u) = max_{x \in M} u(x) - min_{x \in M} u(x).$$

Let $Hameo(M,\omega)$ denote the space of all homeomorphisms $h$ such that there exists a continuous path $\lambda \in PHomeo(M)$ such that

$\lambda(0) = id$, $\lambda(1) = h$

and there exists a Cauchy sequence (for the $d_{ham}$ distance) of hamiltonian isotopies $\Phi_{H^n}$, which $C^0$ converges to $\lambda$ ( in the $\overline{d}$ metric).

The following is the first important theorem in the $C^0$ symplectic topology [11]:

**Theorem (Oh-Muller)**

*The set $Hameo(M,\omega)$ is a topological group. It is a normal subgroup of the identity component $Sympeo_0(M,\omega)$ in $Sympeo(M,\omega)$. If $H^1(M,\mathbb{R}) \neq 0$, then $Hameo(M,\omega)$ is strictly contained in $Sympeo_0(M,\omega)$.*

**Remark**

It is still unkown in general if the inclusion

$$Hameo(M,\omega) \subset Sympeo_0(M,\omega)$$

is strict.

The group $Hameo(M,\omega)$ is the topological analogue of the group $Ham(M,\omega)$ of hamiltonian diffeomorphisms.

The goal of this paper is to construct a subgroup of $Sympeo_0(M,\omega)$, denoted $SSympeo(M,\omega)$ and nicknamed the group of strong symplectic homeomorphisms, containing $Hameo(M,\omega)$, that is:

$$Hameo(M,\omega) \subset SSympeo(M,\omega) \subset Sympeo_0(M,\omega).$$

Like $Hameo(M,\omega)$, the group $SSympeo(M,\omega)$ is defined using a blend of the $C^0$ topology and the Hofer topology on the space $Iso(M,\omega)$ of symplectic isotopies of $(M,\omega)$.

We believe that $SSympeo(M,\omega)$ is "more right" than the group $Sympeo(M,\omega)$ for the $C^0$ symplectic topology. In particular the flux homomorphism seems to exist on $SSympeo(M,\omega)$. This will be the object of a futur paper.

The results of this paper have been announced in [1].

The $C^0$ counter part of the $C^\infty$ contact topology is been worked out in [5], [6].

**2. The symplectic topology** on $Iso(M, \omega)$

Let $Iso(M, \omega)$ denote the space of symplectic isotopies of a compact symplectic manifold $(M, \omega)$. Recall that a symplectic isotopy is a smooth map $H : M \times [0, 1] \to M$ such that for all $t \in [0, 1]$, $h_t : M \to M$, $x \mapsto H(x, t)$ is a symplectic diffeomorphism and $h_0 = id$.

The "Lie algebra" of $Symp(M, \omega)$ is the space $symp(M, \omega)$ of symplectic vector fields, i.e the set of vector fields $X$ such that $i_X\omega$ is a closed form.

Let $\phi_t$ be a symplectic isotopy, then

$$\dot{\phi}_t(x) = \frac{d\phi_t}{dt}(\phi_t^{-1}(x))$$

is a smooth family of symplectic vector fields.

By the theorem of existence and uniqueness of solutions of ODE's,

$$\Phi \in Iso(M, \omega) \mapsto \dot{\phi}_t$$

is a 1-1 correspondence between $Iso(M, \omega)$ and the space $C^\infty([0, 1], symp(M, \omega))$ of smooth families of symplectic vector fields. Hence any distance on $C^\infty([0, 1], symp(M, \omega))$ gives rise to a distance on $Iso(M, \omega)$.

**An intrinsic topology on the space of symplectic vector fields.**

We define a norm $||.||$ on $symp(M, \omega)$ as follows: first we fix a riemannian metric $g$ (which may be the one we used to define $d_0$ above, or any other riemannian metric), and a basis $\mathcal{B} = \{h_1, .., h_k\}$ of harmonic 1-forms.For Hodge theory, we refer to [12].

Recall that the space $harm^1(M, g)$ of harmonic 1-forms is a finite dimensional vector space and its dimension is the first Betti number of $M$.

On $harm^1(M, g)$, we put the following "Euclidean" norm:

for $H \in harm^1(M, g)$ , $H = \sum \lambda_i h_i$, define:

$$|H|_{\mathcal{B}} =: \sum |\lambda_i|.$$

This norm is equivalent to any other norm. Here we choose this one for convenience in the calculations and estimates to come later.

Given $X \in sym(M, \omega)$, we consider the Hodge decomposition of $i_X\omega$ [10] : there is a unique harmonic 1-form $H_X$ and a unique function $u_X$ such that

$$i_X\omega = H_X + du_X$$

Now we define a norm $||.||$ on the the space $symp(M, \omega)$ by:

$$||X|| = |H_X|_{\mathcal{B}} + osc(u_X). \tag{1}$$

It is easy to see that this is a norm. Let us just verify that $||X|| = 0$ implies that $X = 0$. Indeed $|H_X|_{\mathcal{B}} = 0$ implies that $i_X\omega = du_X$, and $osc(u_X) = 0$ implies that $u_X$ is a constant, therefore $du_X = 0$.

**Remark**

This norm is not invariant by $Symp(M, \omega)$. Hence it does not define a Finsler metric on $Symp(M, \omega)$.

The norm $||.||$ defined above depends of course on the riemannian metric $g$ and the basis $\mathcal{B}$ of harmonic 1-forms. However, we have the following:

**Theorem 1**

*All the norms $||.||$ defined by equation (1) using different riemannian metrics and different basis of harmonic 1-forms are equivalent.*

*Hence the topology on the space $symp(M, \omega)$ of symplectic vector fields defined by the norm (1) is intrinsic : it is independent of the choice of the riemannian metric $g$ and of the basis $\mathcal{B}$ of harmonic 1-forms.*

For each symplectic isotopy $\Phi = (\phi_t)$, consider the Hodge decomposition of $i_{(\dot{\phi}_t)}\omega$

$$i_{(\dot{\phi}_t)}\omega = \mathcal{H}_t^\Phi + du_t^\Phi$$

where $\mathcal{H}_t^\Phi$ is a harmonic 1-form.

We define the length $l(\Phi)$ of the isotopy $\Phi = (\phi_t)$ by:

$$l(\Phi) = \int_0^1 (|\mathcal{H}_t^\Phi| + osc(u_t^\Phi))dt = \int_0^1 ||\dot{\phi}_t||dt$$

One also writes

$$\int_0^1 ||\dot{\phi}_t||dt = |||\dot{\phi}_t|||.$$

In the expressions above, we have written $|\mathcal{H}_t^\Phi|$ for $|\mathcal{H}_t^\Phi|_\mathcal{B}$, where $\mathcal{B}$ is a fixed basis of $harm^1(M, g)$, for a fixed riemannian metric $g$.

We define the distance $D_0(\Phi, \Psi)$ between two symplectic isotopies $\Phi = (\phi_t)$ and $\Psi = (\psi_t)$ by:

$$D_0(\Phi, \Psi) = |||\dot{\phi}_t - \dot{\psi}_t||| =: \int_0^1 (|\mathcal{H}_t^\Phi - \mathcal{H}_t^\Psi| + osc(u^{\Phi_t} - u^{\Psi_t}))dt.$$

Denote by $\Phi^{-1} = (\phi_t^{-1})$ and by $\Psi^{-1} = (\psi_t^{-1})$ the inverse isotopies.

**Remarks**

1. The distance $D_0(\Phi, \Psi) \neq l(\Psi^{-1}\Phi)$ unless $\Psi$ and $\Phi$ are hamiltonian isotopies ( see proposition 1).

2. $l(\Phi) \neq l(\Phi^{-1})$ unless $\Phi$ is hamiltonian.

In view of the remarks above, we define a more "symmetrical" distance $D$ by:

$$D(\Phi, \Psi) = (D_0(\Phi, \Psi) + D_0(\Phi^{-1}, \Psi^{-1}))/2$$

Following [11], we define the *symplectic distance* on $Iso(M, \omega)$ by:

$$d_{symp}(\Phi, \Psi) = \overline{d}(\Phi, \Psi) + D(\Phi, \Psi).$$

**Definition**. The *symplectic topology* on $Iso(M, \omega)$ is the metric topology defined by the distance $d_{symp}$.

**Theorem 1'**

*The symplectic topology on $Iso(M,\omega)$ natural : it is independent of all choices involved in its definition.*

We may also define another distance $D^\infty$ on $Iso(M,\omega)$ :

$$D_0^\infty(\Phi,\Psi) = sup_{t\in[0,1]}(|\mathcal{H}_t^\Phi - \mathcal{H}_t^\Psi| + osc(u^{\Phi_t} - u^{\Psi_t}))$$

$$D^\infty(\Phi,\Psi) = ((D_0^\infty(\Phi,\Psi) + D_0^\infty(\Phi^{-1},\Psi^{-1}))/2$$

and

$$d_{symp}^\infty(\Phi,\Psi) = \overline{d}(\Phi,\Psi) + D^\infty(\Phi,\Psi)$$

**Proposition 1.**

*Let $\Phi = (\phi_t), \Psi = (\psi_t)$ be two hamiltonian isotopies and $\sigma_t = (\psi_t)^{-1}\phi_t$ then*

$$|||\dot{\sigma}_t||| = |||\dot{\phi}_t - \dot{\phi}_t||| = \int_0^1 osc(u_t^\Phi - u^{\Psi_t})dt$$

**Proof**

This follows immediately from the equation

$$\dot{\sigma}_t = (\psi_t^{-1})_*(\dot{\phi}_t - \dot{\phi}_t),$$

which is a consequence of proposition 4.                                   □

**Corollary.**

*The distance $d_{sym}$ reduces to the hamiltonian distance $d_{ham}$ when $\Phi$ and $\Psi$ are hamiltonian isotopies.*

The *symplectic topology* reduces to the "hamiltonian topology" of [11] on paths in $Ham(M,\omega)$.

**A Hofer-like metric on $Symp(M,\omega)$**

For any $\phi \in Symp(M, \omega)$, define:

$$e_0(\phi) = inf(l(\Phi))$$

where the infimum is taken over all symplectic isotopies $\Phi$ from $\phi$ to the identity.
The following result was proved in [2].

**Theorem.**

*The map $e : Symp(M, \omega) \rightarrow \mathbb{R} \cup \{\infty\} :$*

$$e(\phi) =: (e_0(\phi) + e_0(\phi^{-1}))/2$$

*is a metric on the identity component $Symp(M, \omega)_0$ in the group $Symp(M, \omega)$,
i.e. it satisfies (i) $e(\phi) \geq 0$ and $e(\phi) = 0$ iff $\phi$ is the identity.*

*(ii) $e(\phi) = e((\phi)^{-1})$*

*(iii) $e\phi.\psi) \leq (e\phi) + e(\psi)$.*

*The restriction to $Ham(M, \omega)$ is bounded from above by the Hofer norm.*

Recall that the Hofer norm [8] of a hamiltonian diffeomorphism $\phi$ is

$$||\phi||_H = inf(l(\Phi_H))$$

where the infimum is taken over all hamiltonian isotopies $\Phi_H$ from $\phi$ to the identity.

The Hofer-like metric above depends on the choice of a riemannian metric $g$ and
a basis $\mathcal{B}$ of harmonic 1-forms. Hence it is not "natural". However, by theorem
1, all the metrics constructed that way are equivalent; so they define a natural
topology on $Symp(M, \omega)$.

## 3. Strong symplectic homeomorphisms

**Definition** : *A homeomorphism $h$ is said to be a strong symplectic homeo-
morphism if there exists a continuous path $\lambda : [0, 1] \rightarrow Homeo(M)$ such that
$\lambda(0) = id; \lambda(1) = h$ and a sequence $\Phi^n = (\phi_t^n)$ of symplectic isotopies, which*

*converges to $\lambda$ in the $C^0$ topology ( induced by the norm $\overline{d}$) and such that $\Phi^n$ is Cauchy for the metric $d_{symp}$.*

We will denote by $SSympeo(M, \omega)$ the set of all strong symplectic homeomorphisms. This set is well defined independently of any riemannian metric or any basis of harmonic 1-forms.

Clearly, if $M$ is simply connected, the set $SSympeo(M, \omega)$ coincides with the group $Hameo(M, \omega)$.

We denote by $SSympeo(M, \omega)^\infty$ the set defined like in $SSympeo(M, \omega)$ but replacing the norm $d_{symp}$ by the norm $d_{symp}^\infty$.

Let $\mathcal{P}Homeo(M)$ be the set of continuous paths $\gamma : [0, 1] \to Homeo(M)$ such that $\gamma(0) = id$, and let $\mathcal{P}^\infty(Harm^1(M)$ be the space of smooth paths of harmonic 1-forms.

We have the following maps:

$A_1 : Iso(M, \omega) \to \mathcal{P}Homeo(M), \Phi \mapsto \Phi(t)$

$A_2 : Iso(M, \omega) \to \mathcal{P}^\infty(Harm^1(M), \Phi \mapsto \mathcal{H}_t^\Phi$

$A_3 : Iso(M, \omega) \to C^\infty(M \times [0, 1], \mathbb{R}), \Phi \mapsto u^\Phi$

Let $\mathcal{Q}$ be the image of the mapping $A = A_1 \times A_2 \times A_3$ and $\overline{\mathcal{Q}}$ the closure of $\mathcal{Q}$ inside $\mathcal{I}(M, \omega) =: \mathcal{P}Homeo(M) \times \mathcal{P}^\infty(Harm^1(M) \times C^\infty(M \times [0, 1], \mathbb{R})$, with the symplectic topology, which is the $C^0$ topology on the first factor and the metric topology from $D$ on the second and third factor.

Then $SSympeo(M, \omega)$ is just the image of the evaluation map of the path at t= 1 of the image of the projection of $\mathcal{Q}$ on the first factor. This defines a surjective map:

$$a : \mathcal{Q} \to SSympeo(M, \omega)$$

The symplectic topology on $SSympeo(M, \omega)$ is the quotient topology induced by $a$.

Our main result is the following

**Theorem 2.**

*Let $(M, \omega)$ be a closed symplectic manifold. Then $SSympeo(M, \omega)$ is an arcwise connected topological group, containing $Hameo(M, \omega)$ as a normal subgroup, and contained in the identity component $Sympeo_0(M, \omega)$ of $Sympeo(M, \omega)$.*

*If $M$ is simply connected, $SSympeo(M, \omega) = Hameo(M, \omega)$. Finally, the commutator subgroup $[SSympeo(M, \omega), SSympeo(M, \omega)]$ of $SSympeo(M, \omega)$ is contained in $Hameo(M, \omega)$.*

**Conjectures**

1. Let $(M, \omega)$ be a closed symplectic manifold, then

$$[SSympeo(M, \omega), SSympeo(M, \omega)] = Hameo(M, \omega).$$

2. The inclusion $SSympeo(M, \omega) \subset Sympeo_0(M, \omega)$ is strict.

3. The results in theorem 2 hold for $SSympeo(M, \omega)^{\infty}$.

Conjecture 3 is supported by a result of Muller asserting that $Hameo(M, \omega)$ coincides with $Hameo(M, \omega)^{\infty}$ which is defined by replacing the $L^{(1,\infty)}$ Hofer norm by the $L^{\infty}$ norm [8].

**Measure preserving homeomorphisms**

On a symplectic $2n$ dimensional manifold $(M, \omega)$, we consider the measure $\mu_{\omega}$ defined by the Liouville volume $\omega^n$. Let $Homeo_0^{\mu_{\omega}}(M)$ be the identity component in the group of homeomorphisms preserving $\mu_{\omega}$. We have:

$$Sympeo_0(M, \omega) \subset Homeo_0^{\mu_{\omega}}(M).$$

Oh and Muller [11] have observed that $Hameo(M, \omega)$ is a sub-group of the kernel of Fathi's mass-flow homomorphism [7]. This is a homomorphism $\theta : Homeo_0^{\mu_{\omega}}(M) \to H_1(M, \mathbb{R})/\Gamma$, where $\Gamma$ is some sub-group of $H_1(M, \mathbb{R})$. Fathi proved that if the dimension of $M$ is bigger than 2, then $Ker\theta$ is a simple group. This leaves open the following question [11]:

*Is $Homeo_0^{\mu_\omega}(S^2) = Sympoe_0(S^2, \omega)$ a simple group?*

But $Sympoe_0(S^2, \omega)$ contains $Hameo(S^2, \omega)$ as a normal subgroup. The question is to decide if the inclusion

$$Hameo(S^2, \omega) \subset Sympoe_0(S^2, \omega)$$

is strict. Since $SSympeo(S^2, \omega) = Hameo(S^2, \omega)$, our conjecture 2 implies that $Homeo_0^{\mu_\omega}(S^2) = Sympoe_0(S^2, \omega)$ is not a simple group, a conjecture of [9].

### Questions

1. Is $SSympeo(M, \omega)$ a normal subgroup of $Sympeo_0(M, \omega)$?
2. Is $[Sympeo_0(M, \omega), Sympeo_0(M, \omega)]$ contained in $Hameo(M, \omega)$?

### 4. Proofs of the results

### 4.1. Proof of theorem 1

If $\mathcal{B}$ and $\mathcal{B}'$ are two basis of $harm^1(M, g)$, then elementary linear algebra shows that $|.|_\mathcal{B}$ and $|.|_{\mathcal{B}'}$ are equivalent. This implies that the corresponding norms on $symp(M, \omega)$ are also equivalent.

Let us now start our construction with a riemannian metric $g$ and a basis $\mathcal{B} = (h_1, ..h_k)$ of $harm^1(M, g)$. We saw that for any $X \in symp(M, \omega)$,

$$i_X \omega = H_X + du_X$$

and we wrote $H_X = \sum \lambda_i h_i$.

Let $g'$ be another riemannian metric. The $g'$-Hodge decomposition of $i_X \omega$ is:

$$i_X \omega = H'_X + du'_X$$

where $H'_X$ is $g'$-harmonic.

Consider the $g'$- Hodge decompositions of the members $h_i$ of the basis $\mathcal{B}$ i.e.

$$h_i = h'_i + dv_i$$

where $h_i'$ is $g'$ harmonic.

$\mathcal{B}' = (h_1', ..h_k')$ is a basis of $harm^1(M, g')$. Indeed if $\sum r_i h_i' = 0$, then $\sum r_i h_i = d(\sum r_i v_i)$. Hence $\sum r_i h_i$ is identically zero because it is an exact harmonic form. Therefore all $r_i$ are zero since $\{h_i\}$ form a basis.

The 1-form

$$H_X'' =: \sum \lambda_i h_i'$$

is a $g'$- harmonic form representing the cohomology class of $i_X \omega$. By uniqueness, $H_X' = H_X''$.

Hence

$$|H_X'|_{\mathcal{B}'} = \sum |\lambda_i| = |H_X|_{\mathcal{B}}$$

Furthermore $H_X' = \sum \lambda_i (h_i - dv_i) = H_X + dv$ where $v = -\sum \lambda_i v_i$. Hence

$$i_X \omega = H_X' + du_X' = H_X + d(v + u_X')$$

By uniqueness in the $g$-Hodge decomposition of $i_X \omega$,

$$u_X = v + u_X'.$$

Denote by $||X||_{g'}$, resp. $||X||_g$, the norm of $X$ using the riemannian metric $g'$ and the basis $\mathcal{B}'$, resp. using the riemannian metric $g$ and the basis $\mathcal{B}$. Then:

$$||X||_{g'} = |H_X'|_{\mathcal{B}'} + osc(u_X') = |H_X'|_{\mathcal{B}'} + osc(u_X - v)$$

$$\leq |H_X'|_{\mathcal{B}'} + osc(u_X) + osc(-v)$$

$$= |H_X|_{\mathcal{B}} + osc(u_X) + osc(v) = ||X||_g + osc(v).$$

Similarly,

$$||X||_g = |H_X|_{\mathcal{B}} + osc(u_X) = |H_X|_{\mathcal{B}} + osc(v + u_X')$$

$$\leq (|H_X|_{\mathcal{B}} + osc(u_X')) + osc(v) = ||X||_{g'} + osc(v).$$

Setting $a = ||X||_g, b = ||X||_{g'}, c = osc(v)$, we proved $a \le b + c$ and $b \le a + c$. Substracting these inequalities, we get $a - b \le b - a$ and $b - a \le a - b$. This gives $a \le b$ and $b \le a$, i.e $a = b$.

We proved that given the couple $(g, \mathcal{B})$ of a riemannian metric $g$ and a basis of $g$-harmonic 1-forms , and any other riemannian metric $g'$, there is a basis $\mathcal{B}'$ of $g'$-harmonic 1-forms so that $||X||_g = |X||_{g'}$, hence the norm $||.||$ is independent of the riemanian metric up to the equivalence relation due to change of basis. In conclusion, all the norms on $symp(M, \omega)$ given by formula (1) are equivalent. $\square$

For the purpose of the proof of the main theorem, we fix a riemannian metric $g$ and a basis $\mathcal{B} = (h_1, .., h_k)$ of $harm^1(M, g)$. The norm of a harmonic 1-form $H$ will be simply denoted $|H|$ and the norm of a symplectic vector field $X$ will be simply denoted $||X||$.

### 4.2. Proof of theorem 2

Let $h_i \in SSympeo(M, \omega)$ $i = 1, 2$ and let $\lambda_i$ be continuous paths in $Homeo(M)$ with $\lambda_i(0) = id$, $\lambda_i(1) = h_i$ and let $\Phi_i^n$ be $d_{symp}$ - Cauchy sequences of symplectic isotopies converging $C^0$ to $\lambda_i$.

Then $\Phi_1^n.(\Phi_2^n)^{-1}$ converges $C^0$ to the path $\lambda_1(t)(\lambda_2(t))^{-1}$. Here $\Phi_1^n.(\Phi_2^n)^{-1}(t) = \phi_1^n(t).(\phi_2^n(t))^{-1}$.

By definition of the distance $d_{symp}$, $\Phi^n$ is a $d_{symp}$ - Cauchy sequence if and only if both $\Phi^n$ and $(\Phi^n)^{-1}$ are $D_0$ - Cauchy and $\overline{d}$- Cauchy sequences.

**Main lemma.**

*If $\Phi^n = (\phi_t^n)$ and $\Psi_t^n = (\psi_t^n)$ are $d_{symp}$ - Cauchy sequences in $Iso(M)$, so is $\rho_t^n = \phi_t^n \psi_t^n$.*

It will be enough to prove that $\rho_t^n$ is a $D_0$ - Cauchy sequence. Indeed since $(\Phi^n)^{-1}$ and $(\Psi^n)^{-1}$ are $D_0$ - Cauchy by assumption, the main lemma applied to their product implies that their product is also $D_0$ Cauchy. Hence $(\Psi^n)^{-1}(\Phi^n)^{-1} =$

$(\Phi^n \Psi^n)^{-1} = (\rho_t^n)^{-1}$ is a $D_0$ - Cauchy sequence. This will conclude the proof that $SSympeo(M, \omega)$ is a group.

We will use the following estimate:

**Proposition 2.** *There exists a constant $E$ such that for any $X \in symp(M, \omega)$, and $H \in harm^1(M, g)$*

$$|H(X)| =: sup_{x \in M} |H(x)(X(x))| \leq E||X||.|H|$$

*Proof.* Let $(h_1, .., h_r)$ be the chosen basis for harmonic 1-forms and let $E = max_i E_i$ and $E_i = sup_V(sup_{x \in M} |h_i(x)(V(x))|$ where $V$ runs over all symplectic vector fields $V$ such that $||V|| = 1$.

Without loss of generality, we may suppose $X \neq 0$ and set $V = X/||X||$. Let $H = \sum \lambda_i h_i$. Then $H(X) = ||X|| \sum \lambda_i h_i(V)$. Hence

$$|H(X)| \leq ||X|| \sum |\lambda_i| sup_x(|h_i(x)(V)(x)|) \leq ||X|| \sum |\lambda_i| E = E||X||.|H|.$$

$\square$

We will also need the following standard facts:

**Proposition 3.**

*Let $\phi$ be a diffeomorphism, $X$ a vector field and $\theta$ a differential form on a smooth manifold $M$, Then*

$$(\phi^{-1})^*[i_X \phi^* \theta] = i_{\phi_* X} \theta$$

**Proposition 4.**

*If $\phi_t, \psi_t$ are any isotopies, and if we denote by $\rho_t = \phi_t \psi_t$, and by $\underline{\phi}_t = (\phi)_t^{-1}$ then*

$$\dot{\rho}_t = \dot{\phi}_t + (\phi_t)_* \dot{\psi}_t$$

*and*

$$\dot{\underline{\phi}}_t = -((\phi)_t^{-1})_* (\dot{\phi}_t)$$

**Proposition 5.**

Let $\theta_t$ be a smooth family of closed 1-forms and $\phi_t$ an isotopy, then

$$\phi_t^* \theta_t - \theta_t = dv_t$$

where

$$v_t = \int_0^t (\theta_t(\dot\phi_s) \circ \phi_s) ds$$

**Proof of the main lemma**

If $\phi_t, \psi_t$ are symplectic isotopies, and if $\rho_t = \phi_t \psi_t$, propositions 3, 4 and 5 give:

$$i(\dot\rho_t)\omega = \mathcal{H}_t^\Phi + \mathcal{H}_t^\Psi + dK(\Phi, \Psi)$$

where $K = K(\Phi, \Psi) = u_t^\Phi + (u_t^\Psi) \circ (\phi_t)^{-1} + v_t(\Phi, \Psi)$, and

$$v_t(\Phi, \Psi) = \int_0^t (\mathcal{H}_t^\Psi(\underline{\dot\phi}_s) \circ \phi_s^{-1}) ds.$$

Let now $\phi_t^n, \psi_t^n$ be Cauchy sequences of symplectic isotopies, and consider the sequence $\rho_t^n = \phi_t^n \psi_t^n$.

We have:

$$|||\dot\rho_t^n - \dot\rho_t^m||| = \int_0^1 |\mathcal{H}_t^{\Phi^n} - \mathcal{H}_t^{\Phi^m} + \mathcal{H}_t^{\Psi^n} - \mathcal{H}_t^{\Psi^m}| + osc(K(\Phi^n, \Psi^n) - K(\Phi^m, \Psi^m))dt$$

$$\leq \int_0^1 |\mathcal{H}_t^{\Phi^n} - \mathcal{H}_t^{\Phi^m})|dt + \int_0^1 |\mathcal{H}_t^{\Psi^n} - \mathcal{H}_t^{\Psi^m})|dt$$

$$+ \int_0^1 osc(u_t^{\Phi^n} - u_t^{\Phi^m})dt + \int_0^1 osc(u_t^{\Psi^n}) \circ (\phi_t^n)^{-1} - u_t^{\Psi^m} \circ (\phi_t^m)^{-1})dt$$

$$+ \int_0^1 osc(v_t(\Phi^n, \Psi^n) - v_t(\Phi^m, \Psi^m)dt$$

$$= |||\dot\phi^n{}_t - \dot\phi^m{}_t||| + \int_0^1 |\mathcal{H}_t^{\Psi^n} - \mathcal{H}_t^{\Psi^m})|dt + A + B$$

where

$$A = \int_0^1 osc(u_t^{\Psi^n}) \circ (\phi_t^n)^{-1} - u_t^{\Psi^m} \circ (\phi_t^m)^{-1}) dt$$

and

$$B = \int_0^1 osc(v_t(\Phi^n, \Psi^n) - v_t(\Phi^m, \Psi^m)) dt$$

We have:

$$A \le \int_0^1 osc(u_t^{\Psi^n}) \circ (\phi_t^n)^{-1} - u_t^{\Psi^m} \circ (\phi_t^n)^{-1}) dt + \int_0^1 osc(u_t^{\Psi^m}) \circ (\phi_t^n)^{-1} - (u_t^{\Psi^m}) \circ (\phi_t^m)^{-1}) dt$$

$$= \int_0^1 osc(u_t^{\Psi^n} - u_t^{\Psi^m}) dt + C$$

where

$$C = \int_0^1 osc(u_t^{\Psi^m} \circ (\phi_t^n)^{-1} - u_t^{\Psi^m} \circ (\phi_t^m)^{-1}) dt.$$

Hence

$$|||\dot{\rho}_t^n - \dot{\rho}_t^m||| \le |||\dot{\phi}_t^n - \dot{\phi}_t^m|||$$

$$+ \int_0^1 |\mathcal{H}_t^{\Psi^n} - \mathcal{H}_t^{\Psi^m})| dt + \int_0^t osc(u_t^{\Psi^n} - u_t^{\Psi^m}) dt + B + C$$

$$= |||\dot{\phi}_t^n - \dot{\phi}_t^m||| + |||\dot{\psi}_t^n - \dot{\psi}_t^m||| + B + C$$

We now show that $C \to 0$ when $m, n \to \infty$.

**Sub-lemma 1 (reparametrization lemma [11])**

$\forall \epsilon \ge 0, \exists m_0$ *such that*

$$C = \int_0^1 osc(u_t^{\Psi^m} \circ (\phi_t^n)^{-1} - u_t^{\Psi^m} \circ (\phi_t^m)^{-1}) dt =: ||u_t^{\Psi^m} \circ (\phi_t^n)^{-1} - u_t^{\Psi^m} \circ (\phi_t^m)^{-1})|| \le \epsilon$$

*if $m \ge m_0$ and $n$ large enough*

**Remark**

This is the "reparametrization lemma" of Oh-Muller [11] (lemma 3.21. (2)). For the convenience of the reader and further references, we include their proof.

**Proof**

For short, we write $u_m$ for $u_t^{\Psi^m}$ and $\mu_t^n$ for $(\phi_t^n)^{-1}$.

First, there exists $m_0$ large such that $||u_m - u_{m_0}|| \leq \epsilon/3$ for $m \geq m_0$, since $(u_m)$ is a Cauchy sequence for the distance $d(u_n, u_m) = \int_0^1 osc(u_n - u_m)dt$.

Therefore

$$||u_m \circ \mu_t^n - u_m \circ \mu_t^m))|| \leq ||u_m \circ \mu_t^n - u_{m_0} \circ \mu_t^n))|| + ||u_{m_0} \circ \mu_t^n - u_{m_0} \circ \mu_t^m))|| + ||u_{m_0} \circ \mu_t^m - u_m \circ \mu_t^m))||$$

$$= ||u_m - u_{m_0}|| + ||u_{m_0} \circ \mu_t^n - u_{m_0} \circ \mu_t^m))|| + ||u_{m_0} - u_m||$$

$$\leq (2/3)\epsilon + ||u_{m_0} \circ \mu_t^n - u_{m_0} \circ \mu_t^m))|.$$

By uniform continuity of $u_{m_0}$, there exists a positive $\delta$ such that if $\overline{d}(\mu_t^m, \mu_t^n) \leq \delta$, then $\max osc((u_{m_0} \circ \mu_t^n - u_{m_0} \circ \mu_t^m)) \leq \epsilon/6$. Hence $||u_{m_0} \circ \mu_t^n - u_{m_0} \circ \mu_t^m))|| \leq \epsilon/3$ for $n, m$ large. Recall that $\mu_t^n$ is a $\overline{d}$- Cauchy sequence. $\square$

To show that $\dot{\rho}_t^n$ is a Cauchy sequenence, the only thing which is left is to show that $B \to 0$ when $n, m \to \infty$.

Let us denote $v_t(\Phi^n, \Psi^n)$ by $v_t^n$ , $\mathcal{H}_t^{\Psi^n}$ by $\mathcal{H}_n^t$ or $\mathcal{H}_n$ and $(\phi_t^n)^{-1}$ by $\mu_t^n$.

For a function on $M$, we consider the norm

$$|f| = sup_{x \in M}|f(x)|$$

We have:

$$|v_t^n - v_t^m| = |\int_0^t (\mathcal{H}_n(\dot{\mu}_s^n) \circ \mu_s^n - \mathcal{H}_m(\dot{\mu}_s^m) \circ \phi_s^m)ds|$$

$$\leq \int_0^1 |((\mathcal{H}_n - \mathcal{H}_m)(\dot{\mu}_s^n)) \circ \mu_s^n|ds$$

$$+ \int_0^1 |\mathcal{H}_m(\dot{\mu}_s^n - \dot{\mu}_s^m)) \circ \mu_s^m|ds$$

$$+ \int_0^1 |\mathcal{H}_m(\dot{\mu}_s^n) \circ \mu_s^n - \mathcal{H}_m(\dot{\mu}_s^n) \circ \mu_s^m|ds$$

The last integral can be estimated as follows:

$$\int_0^1 |\mathcal{H}_m(\dot{\mu}_s^n) \circ \mu_s^n - \mathcal{H}_m(\dot{\mu}_s^n) \circ \mu_s^m|ds$$

$$\leq \int_0^1 |\mathcal{H}_m(\dot{\mu}_s^n) \circ \mu_s^n - \mathcal{H}_m(\dot{\mu}_s^{n_0}) \circ \mu_s^n|ds \tag{1}$$

$$+ \int_0^1 |\mathcal{H}_m(\dot{\mu}_s^{n_0}) \circ \mu_s^n - \mathcal{H}_m(\dot{\mu}_s^{n_0}) \circ \mu_s^m|ds \tag{2}$$

$$+ \int_0^1 |\mathcal{H}_m(\dot{\mu}_s^{n_0}) \circ \mu_s^m - \mathcal{H}_m(\dot{\mu}_s^n) \circ \mu_s^m|ds \tag{3}$$

for some integer $n_0$.

Proposition 2 gives $E|\mathcal{H}_m|D_0((\Phi^n)^{-1}, (\Phi^{n_0})^{-1}) \leq 2E|\mathcal{H}_m|D((\Phi^n), (\Phi^{n_0})^{-1})$ as an upper bound for (1) and (3).

It also gives the following estimates:

$$\int_0^1 |((\mathcal{H}_n - \mathcal{H}_m)(\dot{\mu}_s^n)) \circ \mu_s^n|ds \leq E|\mathcal{H}_n - \mathcal{H}_m| \int_0^1 ||\dot{\mu}_s^n)||ds$$

$$= E.|\mathcal{H}_n - \mathcal{H}_m|.l((\Phi^n)^{-1})$$

and

$$\int_0^1 |(\mathcal{H}_m(\dot{\mu}_s^n - \dot{\mu}_s^m)) \circ \mu_s^m|ds \leq E.|\mathcal{H}_m| \int_0^1 ||(\dot{\mu}_s^n - \dot{\mu}_s^m)||ds$$

$$= E|\mathcal{H}_m|D_0((\Phi^n)^{-1}, (\Phi^m))^{-1}) \leq 2E|\mathcal{H}_m|D(\Phi^n, \Phi^m).$$

Therefore, we get the following estimate:

$$|v_t^n - v_t^m| \leq E.|\mathcal{H}_n - \mathcal{H}_m|l(\Phi^n)^{-1}) + E|\mathcal{H}_m|2(D(\Phi^n, \Phi^m) + 4D(\Phi^n, \Phi^{n_0})) + G$$

where

$$G = \int_0^1 |\mathcal{H}_m(\dot{\mu}_s^{n_0}) \circ \mu_s^n - \mathcal{H}_m(\dot{\mu}_s^{n_0}) \circ \mu_s^m| ds$$

Since $osc(v_t^n - v_t^m) \leq 2|v_t^n - v_t^m|$, we see that

$$\int_0^1 osc(v_t^n - v_t^m) dt \leq 2(l(\Phi^n)^{-1}) \int_0^1 |\mathcal{H}_n^t - \mathcal{H}_m^t| dt$$

$$+E2(D(\Phi^m, \Phi^n) + 4D(\Phi^n, \Phi^{n_0}) \int_0^1 |\mathcal{H}_m^t| dt) + \int_0^1 G dt$$

We need the following facts:

**Sub-lemma 2 (Reparametrization lemma)**

$\forall \epsilon \geq 0, \exists n_0$ *such that*

$$L = \int_0^1 G dt = \int_0^1 (\int_0^1 |\mathcal{H}_m(\dot{\mu}_s^{n_0}) \circ \mu_s^n - \mathcal{H}_m(\dot{\mu}_s^{n_0}) \circ \mu_s^m| ds) dt \leq \epsilon$$

*for $n \geq n_0$ and $m$ sufficiently large.*

**Proposition 6**

$l((\Phi^n))^{-1}$ and $\int_0^1 |\mathcal{H}_m^t| dt$ are bounded for every $n, m$.

We finish first the estimate for $\int_0^1 osc(v_t^n - v_t^m) dt$ using sub-lemma 2 and proposition 6.

Putting together all the information we gathered, we see that:

$$\int_0^1 osc(v_t^n - v_t^m) dt \leq 2(l(\Phi^n)^{-1}) \int_0^1 |\mathcal{H}_n^t - \mathcal{H}_m^t| dt$$

$$+E(2D(\Phi^m, \Phi^n) + 4D(\Phi^n, \Phi^{n_0}))(\int_0^1 |\mathcal{H}_m^t| dt) + L$$

$$\leq 2l((\Phi^n)^{-1})D(\Phi^n, \Phi^m) + E(2D(\Phi^m, \Phi^n) + 4D(\Phi^n, \Phi^{n_0}) \int_0^1 |\mathcal{H}_m^t| dt + L$$

Therefore:

$$\int_0^1 osc(v_t^n - v_t^m)dt \to 0$$

when $n, m \to \infty$, and $n_0$ is chosen sufficiently large.  This finishes the proof of

the main lemma.                                                                              □

**Proof of proposition 6**

This follows from the estmates:

$$l((\Phi^n)^{-1}) \le D((\Phi^n)^{-1}, \Phi^{n_0}) + l(\Phi^{n_0})$$

and

$$\int_0^1 |\mathcal{H}_m^t|dt \le \int_0^1 |\mathcal{H}_m^t - \mathcal{H}_{n_0}^t|dt + \int_0^1 |\mathcal{H}_{n_0}^t|dt$$

$$\le D(\Phi^m, \Phi^{n_0}) + \int_0^1 |\mathcal{H}_{n_0}^t|dt$$

for any $n_0$.  Hence if $n_0$ is sufficiently large, $l((\Phi^n)^{-1})$ and $\int_0^1 |\mathcal{H}_m^t|dt$ are bounded.

□

**Proof of sub-lemma 2**

$$G = \int_0^1 |\mathcal{H}_m(\dot{\mu}_s^{n_0}) \circ \mu_s^n - \mathcal{H}_m(\dot{\mu}_s^{n_0}) \circ \mu_s^m|ds$$

$$\le \int_0^1 |\mathcal{H}_m(\dot{\mu}_s^{n_0}) \circ \mu_s^n - \mathcal{H}_{m_0}(\dot{\mu}_s^{n_0}) \circ \mu_s^n|ds$$

$$+ \int_0^1 |\mathcal{H}_{m_0}(\dot{\mu}_s^{n_0}) \circ \mu_s^n - \mathcal{H}_{m_0}(\dot{\mu}_s^{n_0}) \circ \mu_s^m|ds$$

$$+ \int_0^1 |\mathcal{H}_{m_0}(\dot{\mu}_s^{n_0}) \circ \mu_s^m - \mathcal{H}_m(\dot{\mu}_s^{n_0}) \circ \mu_s^m|ds$$

for some $m_0$.

Exactly like in the proof of sub-lemma 1

$$G(t,n,m) \leq 2|\mathcal{H}_m^t - \mathcal{H}_{m_0}^t|.l(\Psi^{n_0})^{-1})) + F$$

where

$$F = \int_0^1 |\mathcal{H}_{m_0}(\dot{\mu}_s^{n_0}) \circ \mu_s^n - \mathcal{H}_{m_0}(\dot{\mu}_s^{n_0}) \circ \mu_s^m|ds$$

By uniform continuity of $\mathcal{H}_{m_0}(\dot{\mu}_s^{n_0})$, $F \to 0$ when $n, m \to \infty$ since $\mu_t^n$ is Cauchy.

By similar arguments as in the sub-lemma 1, $G \to 0$ and hence $L \to 0$ when $m, n \to \infty$. □

This concludes the proof of that $SSympeo(M, \omega)$ is a group. □

The fact that it is arcwise connected in the ambiant topology of $Homeo(M)$ is obvious from the definition.

$Hameo(M, \omega)$ is a normal subgroup of $SSympeo(M, \omega)$ since it is normal in $Sympeo(M, \omega)$ [11].

Let $h, g \in SSympeo(M, \omega)$ and let $\Phi^n, \Psi^n$ be symplectic isotopies which form Cauchy sequences and $C^0$ converge to $h, g$. By the main lemma the sequence $\Phi^n.\Psi^n.(\Phi^n)^{-1}(\Phi^n)^{-1}$ is a Cauchy sequence. It obviously converges $C^0$ to the commutator $hgh^{-1}g^{-1} \in SSympeo(M, \omega)$.

It is a standard fact that $\Phi^n.\Psi^n.(\Phi^n)^{-1}(\Phi^n)^{-1}$ is a hamiltonian isotopy.

Indeed let $\phi_t$ and $\psi_t$ be symplectic isotopies, and let $\sigma_t = \phi_t \psi_t \phi_t^{-1} \psi_t^{-1}$, then

$$\dot{\sigma}_t = X_t + Y_t + Z_t + U_t$$

with $X_t = \dot{\phi}_t$ , $Y_t = (\phi_t)_* \dot{\psi}_t$, $Z_t = -(\phi_t \psi_t \phi_t^{-1})_* \dot{\phi}_t$, and $U_t = -(\sigma_t)_* \dot{\psi}_t$.

By proposition 5, $i(X_t + Z_t)\omega$ and $i(Y_t + U_t)\omega$ are exact 1-forms. Hence $\sigma_t$ is a hamiltonnian isotopy.

By proposition 1, the metric $D$ coincides with the one for hamiltonian isotopies. Hence $\Phi^n.\Psi^n.(\Phi^n)^{-1}(\Phi^n)^{-1}$ is a Cauchy sequence for $d_{ham}$. Therefore: $[SSympeo(M, \omega), SSympeo(M, \omega)] \subset Hameo(M, \omega)$.

This finishes the proof of the main result.                                    □

## Appendix

For the convenience of the reader, we give here the proofs of propositons 3, 4, and 5.

### Proof of proposition 3

Let $\theta$ be a p-form, $X$ a vector field and $\phi$ a diffeomorphism. For any $x \in M$ and any vector fields $Y_1, .. Y_{p-1}$, we have:

$$(\phi^{-1})^*[i_X \phi^* \theta](x)(Y_1, ..., Y_{p-1}) = (i_X \phi^* \theta)(\phi^{-1}(x))(D_x \phi^{-1}(Y_1(x)), ...(D_x \phi^{-1}(Y_{p-1}(x)))$$

$$= (\phi^* \theta)(\phi^{-1}(x))(X_{\phi^{-1}(x)}, D_x \phi^{-1}(Y_1(x)), ...(D_x \phi^{-1}(Y_{p-1}(x)))$$

$$= \theta(\phi(\phi^{-1}(x))(D_{\phi^{-1}(x)}\phi(X_{\phi^{-1}(x)}), D_{\phi^{-1}(x)}\phi D_x \phi^{-1}(Y_1(x)), ...D_{\phi^{-1}(x)}\phi D_x \phi^{-1}(Y_{p-1}(x))$$

$$= \theta(x)((\phi_* X)_x, Y_1(x), .. Y_{p-1}(x))$$

$$= (i(\phi_* X)\theta)(x)(Y_1, .., Y_{p-1})$$

since $D_{\phi^{-1}(x)}\phi D_x \phi^{-1} = D_x(\phi \phi^{-1}) = id.$

Therefore $(\phi^{-1})^*[i_X \phi^* \theta] = i(\phi_* X))\theta$

                                                                              □

### Proof of proposition 4

This is just the chain rule. See [6] page 145.                                 □

### Proof of proposition 5

For a fixed $t$, we have

$$\frac{d}{ds}\phi_s^* \theta_t = \phi_s^*(L_{\dot{\phi}_s}\theta_t,$$

where $L_X$ is the Lie derivative in the direction $X$. Since $\theta$ is closed, we have:

$$\frac{d}{ds}\phi_s^* \theta_t = \phi_s^*(di_{\dot{\phi}_s}\theta_t) = d(\phi_s^*(\theta_t(\dot{\phi}_s))) = d(\theta_t(\dot{\phi}_s) \circ \phi_s).$$

Hence for every $u$

$$\phi_u^* \theta_t - \theta_t = \int_0^u \frac{d}{ds}\phi_s^* \theta_t ds = d(\int_0^u (\theta_t(\dot{\phi}_s) \circ \phi_s)ds$$

Now set $u = t$.                                                                                    $\square$

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